Quiver Topology and RG Dynamics

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Abstract

Renormalization group flows of quiver gauge theories play a central role in determining the low-energy properties of string vacua. We demonstrate that useful predictions about the RG dynamics of a quiver gauge theory may be extracted from the global structure of its quiver diagram. For quiver theories of a certain type, we develop an efficient and practical method for determining which superpotential deformations generate a flow to an interacting conformal fixed point.

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1   Introduction

Product gauge theories with bifundamental matter, known as quiver gauge theories, include many well-motivated extensions of the Standard Model. They also arise as effective field theories of open strings living on stacks of D-branes. The renormalization group (RG) flows of quiver gauge theories from stringy scales down to observable energies play a crucial role in determining the low-energy physics associated with a given string vacuum. Thus, our ability to connect string theory usefully to low-energy physics hinges, in part, on how well we can understand the RG dynamics of quiver theories. Such an understanding is a prerequisite to addressing questions like: To what extent is the Standard Model a "typical" endpoint of RG flows from the landscape of string vacua? What TeV-scale extensions of the Standard Model are consistent with string theory and how are they distributed? For example, in [1] it was suggested that statistical universality in a random, but bounded, landscape of vacua might lead to predictions for the high scale effective field theory arising from string theory. But to derive predictions for the *low* energy world in this approach, RG flow of generic quiver theories needs to be understood.

The problem is that even simple quiver theories may have quite complex and unpredictable RG flows. Performing Seiberg dualities [2] on different nodes of the quiver [3, 4, 5, 6, 7, 8] yields many complex patterns, such as periodic [9] or chaotic [10] duality cascades, as well as structures like duality walls and trees [11]. The infrared limit of a flow may depend sensitively on initial conditions, and IR fixed points may lie on conformal manifolds with many branches [12, 13]. In the absence of general techniques for precisely evaluating the flows of strongly coupled gauge theories, it would be quite useful to find simple criteria that could be used to infer the infrared behavior of a given quiver theory.

The present paper attempts to extract information about the RG dynamics of an $\mathcal{N} = 1$ supersymmetric quiver gauge theory from the global structure of its quiver diagram. To make the problem tractable, we choose to study equal rank quivers with N_c colors and $N_f = 2N_c$ flavors at each node. We choose these quivers because each node is well within the conformal window and thus, when there is no superpotential, the theories flow to interacting conformal fixed points in the infrared. We give a simple and efficient method of determining which superpotentials will drive these theories to new interacting conformal fixed points and which ones will drive unbounded flows in the original couplings. Our results crucially involve the global structure of the quiver and are inaccessible to the sort of node-by-node analysis that has been the basis of most previous studies of quiver dynamics.

2 Physical Preliminaries

We study $\mathcal{N} = 1$ equal rank $\prod SU(N_c)$ quiver gauge theories that are specified by an oriented graph G with nodes $v \in V$ corresponding to gauge groups, and edges $e \in E$ corresponding to bifundamental chiral multiplets Φ_j^i. The theory is free from gauge anomalies if at each node the numbers of incoming and outgoing edges are the same. We will only consider chiral theories since nonchiral fields will acquire masses under generic RG flows and can be integrated out.

2.1 Interacting superconformal fixed points

Our main goal in this paper will be to find a criterion for distinguishing which $\mathcal{N} = 1$ supersymmetric quiver theories flow to interacting superconformal fixed points in the IR. The requirements for such a fixed point are:

1. All the NSVZ β-functions [14] vanish.

2. All operators present in the superpotential are marginal (or irrelevant).

When an assignment of R-charges satisfying these conditions can be found, we will make the assumption that an interacting superconformal fixed point exists. In the context of the quivers we study in this paper, such an assumption is reasonable, although it is not true in general that these conditions are sufficient for the existence of a conformal fixed point. If a family of such solutions exists, the R-charge appearing in the superconformal algebra can be found via a-maximization [15]. When a simultaneous solution to conditions 1 and 2 does not exist, we will see that the theory is pushed toward arbitrarily strong coupling. In what follows, we will refer to the requirements implied by condition 2 as the "marginality inequalities."

In addition to conditions 1 and 2, there is also a unitarity requirement. Given a gauge-invariant operator $\mathcal{O}$, superconformality requires that its dimension $\Delta(\mathcal{O})$ satisfies $\Delta(\mathcal{O}) \geq 1$. In our theories, this implies that the R-charges of all fields must be positive. To see this, consider the gauge-invariant dibaryon operator

$$B_\Phi = \epsilon_{i_1 i_2 \dots i_{N_c}} \epsilon^{j_1 j_2 \dots j_{N_c}} \Phi_{j_1}^{i_1} \Phi_{j_2}^{i_2} \dots \Phi_{j_{N_c}}^{i_{N_c}}. \tag{1}$$

Since this operator is chiral, it satisfies $R(B_\Phi) = N_c R(\Phi)$. In the large N_c limit, $R(\Phi) < 0$ will make this operator dramatically violate the unitarity bound. Note that $R(\Phi) = 0$ also implies a violation this bound. Accordingly, at a superconformal fixed point it must also be true that

$$R(e) > 0 \quad \forall e \in E. \tag{2}$$

If a solution to conditions 1 and 2 implies violations of the unitarity bound, we regard this as a sign that new accidental symmetries have appeared in the infrared.

A particularly convenient class of quiver gauge theories for our purpose has nodes which all have equal ranks N_c and satisfy $N_f = 2N_c$. When the superpotential vanishes ($W = 0$) for such a quiver, it is reasonable to suppose that its RG flow reaches a superconformal fixed point starting from generic initial conditions in the UV. This is because the assumption that $N_f = 2N_c$ at each node places us firmly in the conformal window $3N_c/2 < N_f < 3N_c$ [2], and while it is possible that internodal dynamics in some cases might push some gauge couplings to IR freedom, we know of no reason to believe that this happens here. The R-charge of each bifundamental at the $W = 0$ fixed point will be

$$R(e) = 1 - \frac{N_c}{N_f} = 1/2 \,, \tag{3}$$

which follows from the vanishing of the NSVZ β-functions.

In this paper our approach will be to start with the theory at the $W = 0$ interacting fixed point and ask what happens when it is deformed by the addition of relevant operators. By eq. (3), quartic and higher order operators have $R(\mathcal{O}) \geq 2$ and are irrelevant. Therefore, in what follows we focus attention on theories which contain cubic operators (mass terms are ruled out by the assumption of chirality).

Quiver theories of this kind with $W = 0$ have a moduli space of vacua parameterized by VEVs of the matter fields that solve the D-term equations. At a generic point in this moduli space the gauge symmetry is broken. For simplicity, we will focus attention on the origin in moduli space where the VEVs vanish and the gauge symmetry is unbroken.

2.2 Flows and β-functions

All flows in the theory are driven by the β-functions. The flow of the gauge couplings is described by the NSVZ β-function [14]

$$\beta_{1/g^2} = \frac{3T(G) - \sum_a T(r_a)(1 - \gamma_a(g))}{8\pi^2 - g^2 T(G)} \,, \tag{4}$$

where $\operatorname{Tr} T_{r_a}^A T_{r_a}^B = \frac{1}{2} T(r_a) \delta^{AB}$, the sum is over all matter fields Φ_a in the theory, and γ_a is the anomalous dimension of Φ_a. For equal-rank quivers with $N_f = 2N_c$ at each node, the NSVZ beta function for the gauge coupling at node v is

$$\beta_{1/g_v^2} = \frac{N_c}{8\pi^2 - g_v^2 N_c} \left(1 + \frac{1}{2} \sum_{e \sim v} \gamma_e(\{g_i\}) \right) , \tag{5}$$

where $e \sim v$ denotes all edges entering or exiting the node. We use the normalization $T(SU(N_c)) = N_c$ and $T(\square) = \frac{1}{2}$.

In the vicinity of a superconformal fixed point we can use

$$1 + \frac{1}{2}\gamma_e = \Delta(e) = \frac{3}{2}R(e) \tag{6}$$

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to express the beta functions in terms of $R(e)$. Slightly away from a superconformal point, the R-charges on the right hand side can be defined according to the formalism of [16]. Using (6), one can then write

$$\beta_{1/g_v^2} \propto \sum_{e \sim v} R(e) - 2 \tag{7}$$

with a positive coefficient of proportionality. Although the coefficient includes a potentially dangerous denominator, we will assume that we stay far away from poles in the NSVZ β-function.

Similarly, given a superpotential term $\int d^2\theta\, \lambda_{\mathcal{O}} \mathcal{O}$, the beta function for the coupling $\lambda_{\mathcal{O}}$ is

$$\beta_{\lambda_{\mathcal{O}}} \propto \sum_{e \in \mathcal{O}} R(e) - 2 \equiv R(\mathcal{O}) - 2 \;, \tag{8}$$

with a positive coefficient of proportionality. Irrelevant operators are those with $R(\mathcal{O}) > 2$.

3 Examples of RG Dynamics

Before investigating arbitrary $N_f = 2N_c$ quivers we will study several examples of theories in this class. The examples are chosen to illustrate cases when a suitable choice of superpotential forces a superconformal interacting theory to flow to strong coupling. The main result of this paper, derived in Sec. 5, is a general characterization of superpotentials which have this effect.

3.1 The octahedral theory

Consider the vertices of the octahedron as nodes of a quiver diagram, with edges corresponding to bifundamentals. There exists exactly one assignment of arrows (up to charge conjugation) which ensures that each of the eight faces of the octahedron forms a closed loop, and thus is an eligible superpotential term. We call this theory the octahedral theory; see Fig. 1. Note that the outer triangle is also a face of the octahedron and is a closed loop appearing in the superpotential.

It is easy to see that the faces of the octahedron cannot all simultaneously be marginal or irrelevant if the theory is conformal. This is because the conformality condition on the gauge coupling of the node v reads:

$$\sum_{e \sim v} R(e) = 2 \tag{9}$$

so that, summing over all vertices, we get:

$$\sum_{e \in E} R(e) = \frac{1}{2} \sum_{v \in V} \sum_{e \sim v} R(e) = \frac{1}{2} \sum_{v \in V} 2 = 6 \;. \tag{10}$$
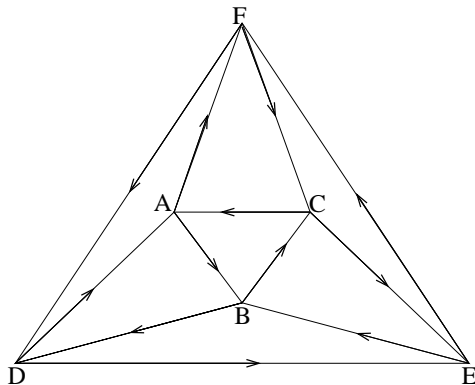
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Figure 1: The octahedral quiver.

The pre-factor $1/2$ accounts for the fact that a sum over nodes counts every bifundamental exactly twice. Likewise, if we assume that each face is marginal or irrelevant, we can re-do the same calculation face by face and arrive at:

$$\sum_{e \in E} R(e) = \frac{1}{2} \sum_{f \in F} \sum_{e \sim f} R(e) \geq \frac{1}{2} \sum_{f \in F} 2 = 8 \,, \tag{11}$$

where the index f runs over the faces of the octagon (cubic terms in the superpotential) and the factor of $1/2$ again accounts for the fact that each bifundamental participates in exactly two loops. Because $6 \ngeq 8$, we see that no superconformal fixed point accommodating all the cubic superpotential terms can exist.

To see which superpotentials are compatible with an interacting superconformal fixed point, start with

$$W_{\text{trial}} = \lambda_1 \, \mathcal{O}_{\text{ABC}} \,, \tag{12}$$

where we choose to label operators (loops in the quiver) by the nodes on which they rest. $\mathcal{O}_{\text{ABC}}$ will drive the flow to the unique R-charge assignment where it is marginal and the NSVZ β-functions vanish:

$$
\begin{aligned}
R(e_{\text{AB}}) = R(e_{\text{BC}}) = R(e_{\text{CA}}) = R(e_{\text{DE}}) = R(e_{\text{EF}}) = R(e_{\text{FD}}) &= 2/3 \\
R(e_{\text{AF}}) = R(e_{\text{FC}}) = R(e_{\text{CE}}) = R(e_{\text{EB}}) = R(e_{\text{BD}}) = R(e_{\text{DA}}) &= 1/3 \,.
\end{aligned} \tag{13}
$$

At this point $\mathcal{O}_{\text{DEF}}$ becomes marginal, too. However, all other cubic operators are relevant and an addition of any one of them to the superpotential will automatically produce an unending RG trajectory. Any pair

$$\{\mathcal{O}_{\text{ABC}}, \mathcal{O}_{\text{any} \, \triangle \neq \text{DEF}}\} \tag{14}$$

is therefore forbidden from appearing simultaneously in the superpotential if one is interested in an interacting superconformal field theory whose low energy degrees of freedom are described by the quiver of Fig. 1.
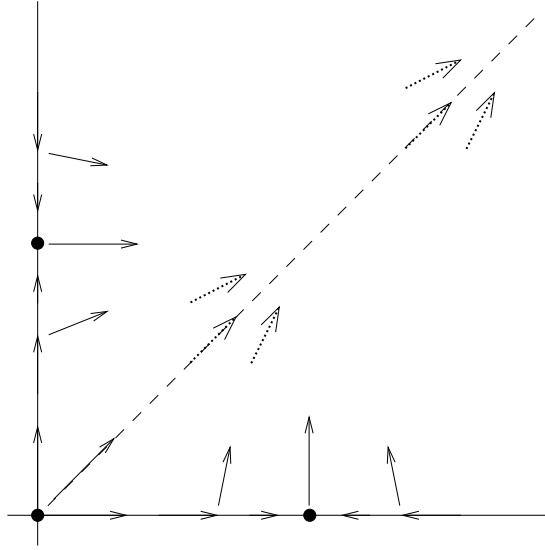
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Figure 2: The flow diagram of the octahedral theory drawn in the space parameterized by two superpotential couplings, which cannot both be turned on at an interacting superconformal fixed point.

To understand the physics better, in Fig. 2 we draw a flow diagram in the superpotential coupling space parameterized by $(\lambda_1)^2$, $(\lambda_2)^2$, with λ_2 corresponding to, say, $\mathcal{O}_{\mathrm{ABD}}$. The argument of the previous paragraph proves that there is an interacting superconformal fixed point when $W = \lambda_1 \mathcal{O}_{\mathrm{ABC}}$, although it is not stable to $\mathcal{O}_{\mathrm{ABD}}$ deformations. This is captured by the arrows pointing along and away from one of the axes. Because of the symmetry of the quiver, an identical set of arrows must be drawn on the other axis, too. The $W = 0$ fixed point with $R(e) \equiv 1/2$ is of course unstable in both directions in superpotential coupling space, as is represented by the arrows pointing away from the origin. Overall, the flow arrows marked with continuous lines can be deduced rigorously from the NSVZ and superpotential β-functions.

The remaining flow vectors, shown as dotted arrows, are expected to point away from the axes. We have already argued that there are no fixed points away from the axes, and we assume that limit cycles in the flow do not occur (although recently discovered counterexamples to the a-theorem leave open the possibility that limit cycles could exist in four dimensions [17, 18]). Thus the flow must carry the theory to infinity.

The more complete treatment of Sec. 5 will reveal that in such a situation, we expect that at least one gauge coupling will run to infinity as well. This means that the IR degrees of freedom are best analyzed by Seiberg dualizing [2] the strongest gauge group. In the octahedral example, dualizing any node [3] gives the quiver of Fig. 3. The four nodes on the horizontal axis of the diagram have $N_f = N_c$ and are expected to confine, producing the same diagram as occurs in the conifold theory [19].
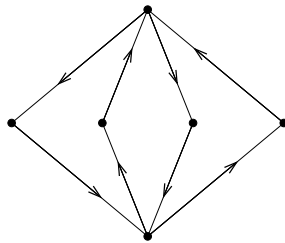
Figure 3: The octahedral theory after one Seiberg duality.

The confinement, which becomes apparent only in a Seiberg dual picture, explains why there could not exist a fixed point whose low energy degrees of freedom would be those of Fig. 1.

In general, we will call collections of superpotential terms like eq. (14) which push the theory to strong coupling **forbidden sets**. A set of terms is forbidden if the terms cannot all be made marginal or irrelevant without making the NSVZ β-functions nonvanishing. When a forbidden set is added to the superpotential, the theory *appears to* flow without end to a strong coupling regime. We take this to be a sign of new dynamics, which may become apparent in, e.g., a different Seiberg duality frame. In the remainder of this paper we shall refer to flows resulting from forbidden sets using words like "indefinite flow." This language is meant as a shorthand for a situation where the degrees of freedom characterizing an infrared fixed point are different from those represented by the original quiver diagram.

3.2 A locally characterized (nearly) forbidden set

In Sec. 3.1 the forbidden sets of the octahedral theory (14) were identified after finding the global solution (13). In this subsection, as an example of what we are trying to achieve, we will try to identify a forbidden set using local considerations alone.

Consider a fragment of an $N_f = 2N_c$ equal rank quiver as shown in Fig. 4, and assume that the two triangular loops are present in the superpotential. Let $a, \ldots, g$ label the R-charges of the bifundamental fields. If this quiver and superpotential are to be superconformal, the R-charges must satisfy

$$
\begin{aligned}
\text{I} : \quad & a + b + d + e & = & \quad 2 \\
\text{II} : \quad & b + c + f + g & = & \quad 2 \\
\text{III} : \quad & b + d + g & \geq & \quad 2 \\
\text{IV} : \quad & b + e + f & \geq & \quad 2 \,.
\end{aligned}
\tag{15}
$$

Equations I and II set the NSVZ β-functions to zero while III and IV ensure that the two triangular loops are marginal or irrelevant. Now notice that the linear combination I+II-III-IV reads:
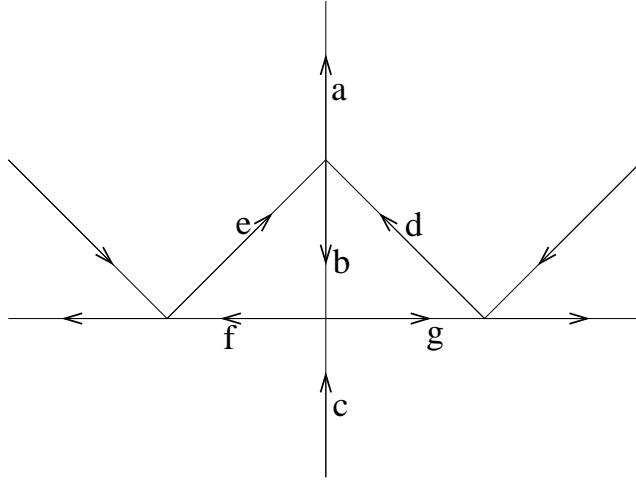
$$
a + c \leq 0 \,,
\tag{16}
$$

Figure 4: A local obstruction to a consistent R-charge assignment.

threatening a violation of the unitarity bound. Indeed, the best we can do is to take $a = c = 0$ in which case the dibaryons built from these two fields have vanishing dimension and hence violate the unitarity bound. Thus, the two triangles of Fig. 4 already threaten to form a forbidden set regardless of what happens in the rest of the quiver. Note that this locally characterized (nearly) forbidden set is present in the octahedral quiver, Fig. 1. More complex examples below will show that forbidden sets cannot always be fully characterized by local data in the quiver.

3.3 There are other forbidden sets

So far we have only seen one example of a forbidden set, namely the setup of Fig. 4. For another example, consider the quiver in Fig. 5. Assume that all gauge invariant terms are present in the superpotential with non-zero couplings.
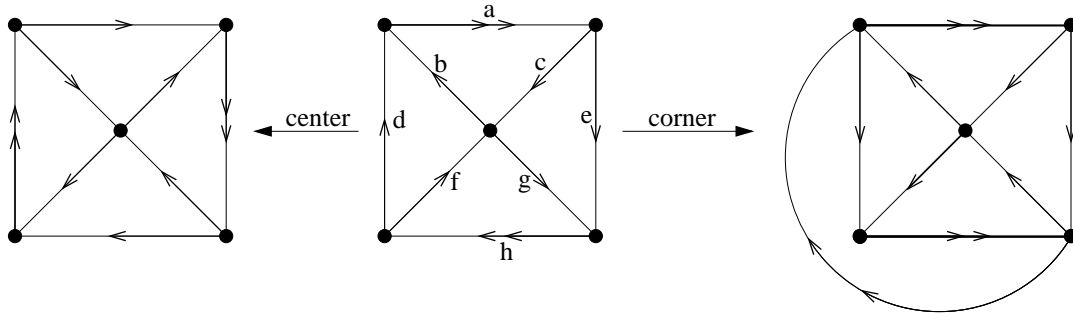

Figure 5: A naïvely superconformal quiver may be Seiberg dual to one that is IR free. On the right Seiberg duality was performed on the bottom left node.

The R-charges must satisfy conditions analogous to eqs. (15):

$$
\begin{array}{rrcl}
\text{I}: & 2a+b+d & = & 2 \\
\text{II}: & 2a+c+e & = & 2 \\
\text{III}: & b+c+f+g & = & 2 \\
\text{IV}: & 2h+d+f & = & 2 \\
\text{V}: & 2h+e+g & = & 2 \\
\text{VI}: & a+b+c & \geq & 2 \\
\text{VII}: & f+g+h & \geq & 2 \\
\text{VIII}: & a+d+e+h & \geq & 2 \,.
\end{array}
\tag{17}
$$

The linear combination

$$
\frac{1}{2}(\text{I}+\text{II}+\text{III}+\text{IV}+\text{V})-(\text{VI}+\text{VII}+\text{VIII}): 0 \leq -1
\tag{18}
$$

reveals that it is impossible to satisfy these conditions simultaneously.

In this case, the operators corresponding to eqs. VI-VIII form a forbidden set. If these terms are simultaneously present in the superpotential, their couplings will grow without end because the operators will never reach marginality. This is another example of an indefinite flow like the dashed line in Fig. 2. In Sec. 5 we shall see that in such a situation at least some subset of the gauge couplings will grow large, too. Therefore, in the infrared a Seiberg dual description of the theory will apply. The Seiberg dual may allow us to identify the new dynamics of which the apparently unending RG trajectory is a sign.

We can either dualize the central node or a corner node. But Seiberg duality on the central node gives back the same quiver, so the only interesting nodes are the corners. The resulting quiver is shown on the right of Fig. 5. In the dual theory, two nodes get $N_f = 3N_c$, which suggests that the new dynamics could be IR free. However, it may also be possible that the other nodes pull the naïvely IR free node back to being interacting, as happens with some string-derived theories.

3.4   A global example

In most cases when a forbidden set is added to the superpotential and the theory is pushed to strong coupling, selecting the correct Seiberg duality frame in which to study the infrared phase is difficult. We illustrate the difficulty of the general problem by presenting a formidable instance of it. Specifically, we consider a family of quivers (defined below), which is parameterized by an integer b. The family is a subset of the class of quivers considered in this paper. Every quiver in the family contains a forbidden set so with a suitable choice of superpotential, each theory is driven to a strong coupling regime. But for all $b > 1$, the methods used in the previous examples are insufficient for picking the right Seiberg dual picture and the tools developed in Sec. 5 are necessary. As a bonus, we shall have more to learn from the limit $b \to \infty$.

We construct our family of examples by assembling building blocks represented in Fig. 6. Each building block is structured around a central triangle, with three batches of lines parallel to the three sides forming an array of surrounding quadrangles. The building blocks may be indexed by the numbers of parallel lines in each batch, so the block in the figure would be denoted $\triangle_{2,3,4}$. Each triangle (quadrangle) in the building block forms a closed loop in the quiver and therefore represents a possible cubic (quartic) term in the superpotential.
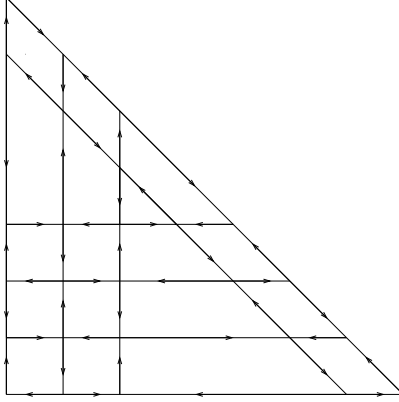

Figure 6: A triangular building block $\triangle_{2,3,4}$.

Consider the octahedral theory of Fig. 1. It is made up of eight building blocks $\triangle_{1,1,1}$ assembled together to form an octahedron. If we now replace each building block with $\triangle_{b,b,b}$, we shall form another quiver diagram with $N_f = 2N_c$ on all nodes. Call the resulting quiver G_b, so the octahedral diagram is G_1. It is easy to see that if all the cubic and quartic terms are turned on in the superpotential, the couplings of G_b run indefinitely. This may be seen using the fact that each G_b can be embedded on a sphere, and therefore

$$|F| = |E| - |V| + \chi = (2|V|) - |V| + \chi = |V| + \chi, \tag{19}$$

where $|E| = 2|V|$ holds because exactly two edges end at each vertex. If all the cubic and quartic operators are turned on, then at a superconformal fixed point the following relations, analogous to eqs. (10-11), must hold:

$$|V| = \frac{1}{2}\sum_{v\in V} 2 = \frac{1}{2}\sum_{v\in V}\sum_{e\sim v} R(e) = \sum_{e\in E} R(e) = \frac{1}{2}\sum_{f\in F}\sum_{e\sim f} R(e) \geq \frac{1}{2}\sum_{f\in F} 2 = |F| = |V| + \chi. \tag{20}$$

Because we are on a sphere, $\chi = 2$ and such a fixed point cannot exist. Therefore, a Seiberg dual description is again necessary to study the theory in the IR, but now, in contrast to the previous examples, there is no tractable method to reveal on which node to dualize. The techniques of Sec. 5 will select a likely candidate.

Importantly, in the present example the indefinite flow of the theory is a direct consequence of the spherical structure of the quiver, $\chi \not\leq 0$. Indeed, in $G_{b\to\infty}$ we can make arbitrarily large local patches of the quiver perfectly superconformal, that is, imposing that the NSVZ β-functions vanish and that all terms in the superpotential are marginal or irrelevant. We may only conclude that the RG flow of the theory with a generic superpotential will never rest after accounting for the fact that the building blocks glue up together to form a sphere. The present example illustrates explicitly that global considerations are a necessary ingredient in a general study of renormalization group flows.

4   Superconformality and Quiver Structure

We will show that necessary conditions for the existence of superconformal fixed points of our gauge theories are encoded in the oriented and unoriented incidence matrices of the quiver.

The unoriented incidence matrix B, which keeps track of which edges connect to which vertices, will turn out to be related to changes in the NSVZ β-functions at each node. This occurs because the β-functions do not discriminate between fundamentals and anti-fundamentals (their quadratic Casimirs are the same), so all one needs to know is which edges hit which nodes, and not whether the edges are entering or exiting. We will show that the kernel of B (ker B) encodes R-charge assignments (vectors in $\mathbb{R}^E$) which leave the NSVZ β-functions unchanged.

In contrast, the oriented incidence matrix A, which includes the direction of connectivity via plus and minus signs, turns out to encode information about superpotential deformations. Because gauge-invariant operators are closed loops, the orientation of the edges is essential for determining which operators are allowed. As we will describe, the kernel of A (ker A) encodes those R-charge assignments which are correlated with the running of superpotential couplings.

Our objective is to scan the space $\mathbb{R}^E$ of candidate R-charge assignments for configurations satisfying the superconformality conditions in Sec. 2. Each possible R-charge assignment is represented as a column vector $\vec{R}^{\text{tot}} \in \mathbb{R}^E$. In the class of theories we study, the NSVZ β-functions vanish when

$$\vec{R}^{\text{tot}} = \vec{r} \equiv \frac{1}{2} \cdot \vec{1}, \tag{21}$$

where $\vec{1} \equiv (1, 1, \ldots, 1)$, i.e., by assigning R-charge $1/2$ to each edge. We then parameterize the possible R-charges at an interacting superconformal fixed point by deviations from $\vec{r}$, taking

$$\vec{R}^{\text{tot}} = \vec{r} + \vec{R}. \tag{22}$$

The role of $\vec{R}$ will be to ensure that the marginality inequalities and the unitarity bound are satisfied while keeping the NSVZ β-functions zero, which is already

accomplished by $\vec{r}$.

4.1   Vanishing of the NSVZ β-functions

The NSVZ β-function [14],

$$\beta_{1/g^2} = \frac{3T(G) - \sum_a T(r_a)(1 - \gamma_a(g))}{8\pi^2 - g^2 T(G)}\,, \tag{23}$$

is necessarily zero at any superconformal fixed point. The numerator of the NSVZ β-function can be usefully recast as the anomaly in the R-current. Thus, satisyfing $\beta_{NSVZ} = 0$ is tantamount to having an R-anomaly free theory; this transpires because the trace of the stress tensor is in the same supermultiplet as the divergence of the R-current. Because of this, any anomaly free global $U(1)$ can be added to a candidate R-symmetry while still satisfying $\beta_{NSVZ} = 0$. Consequently, there will be a vector space of possible solutions to $\beta_{NSVZ} = 0$. In the quiver gauge theories we are considering, there is one global $U(1)$ for each of the $|E|$ edges in the quiver, but anomaly freedom requires that the net added $U(1)$ charge vanish at each of the $|V|$ nodes. This gives $|V|$ linear constraints on $|E| = 2|V|$ variables for our $N_f = 2N_c$ quivers. If the constraints are non-degenerate, this gives a solution space of anomaly-free R-charge assignments of dimension $|V|$. Degeneracies between the anomaly constraints may increase the dimension of the solution space. In this section we will describe a simple way of representing this solution space for our quiver gauge theories.

Recall from eq. (7) that for $N_f = 2N_c$ quivers, the NSVZ β-function for the gauge coupling g_v is given (up to an unimportant positive coefficient) by

$$\beta_{g_v} \propto 2 - \sum_{e \sim v} R^{\mathrm{tot}}(e)\,. \tag{24}$$

To represent this succintly, define the $|V| \times |E|$ incidence matrix B by

$$B_{ve} = \begin{cases} 1 & \text{if edge } e \in E \text{ is incident to vertex } v \in V \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

Then, splitting the total R-charge as in (22), the vanishing of the $|V|$ NSVZ β-functions is written as

$$\beta_{g_v} \propto 2 - \sum_e B_{ve}\left(r^e + R^e\right) = 0 \quad \Longrightarrow \quad B\vec{R} = 0\,, \tag{26}$$

where we used $B_{ve}r^e = 2$. In other words,

$$\vec{R} \in \ker B\,. \tag{27}$$

Equivalently, adding $\vec{R}$ to a candidate R-charge assigment leaves the NSVZ β-functions unchanged because $\vec{R}$ is the charge vector of an anomaly-free flavor symmetry $U(1)_F$. The orthogonal complement

$$(\ker B)^\perp = \mathrm{Im}\, B^{\mathrm{T}} \tag{28}$$

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comprises R-charge assignments which necessarily make the NSVZ β-functions nonzero, hence moving away from a superconformal fixed point.

The kernel of the unoriented incidence matrix B of the quiver thus describes a vector space of possible R-charge assignments at a fixed point of the gauge couplings. What is the dimension of this vector space? The rank of B is known [20] to be $|V| - 1$ if the graph is bipartite and $|V|$ otherwise. (Recall that a bipartite graph is one for which the nodes can be labeled $+$ or $-$ in such a way that edges only connect nodes of opposite sign.) B has reduced rank for a bipartite graph since the sum $\sum_v (-1)^v B_{ve} = 0$, where $(-1)^v$ is the label of the node. Equivalently, a quiver can have at most one degeneracy in its R-anomaly cancellation equations, and this happens if and only if the quiver is bipartite.

In chiral bipartite quivers, gauge invariant operators are at least quartic in the bifundamentals and thus cannot drive a flow from the $W = 0$ fixed point with $\vec{R}^{\text{tot}} = \vec{r}$ at which our RG flows start. For this reason, we shall fix our attention on *non-bipartite* quivers in this paper, giving rank $B = |V|$ and

$$\dim \ker B = |V| = \dim \operatorname{Im} B^{\mathrm{T}}. \tag{29}$$

Here we used the fact that $\dim \ker B + \operatorname{rank} B = |E|$ and $|E| = 2|V|$ because every vertex is the initial vertex of exactly two edges. Thus, we have $|V|$ linearly independent $U(1)_F$ charge vectors that can be added to the R-symmetry at the superconformal point, along with $|V|$ linearly independent ways of altering the NSVZ β-functions. One of the latter modes is the vector of unities $\vec{1}$.

4.2 Marginality of the superpotential

The flows we consider start by the addition of a superpotential containing relevant operators at the $W = 0$ fixed point of a quiver gauge theory. If this theory flows to a new superconformal fixed point, all operators in the superpotential will have to become marginal or irrelevant. Thus the β-function for all the superpotential couplings should satisfy

$$\beta_{\lambda_{\mathcal{O}}} \propto \sum_{e \in \mathcal{O}} R^{\text{tot}}(e) - 2 \geq 0. \tag{30}$$

To achieve this, the R-charges of operators in the superpotential will have to shift from their values at the $W = 0$ fixed point where they are determined by setting $\vec{R}^{\text{tot}} = \vec{r}$ (21), i.e. by assigning an R-charge of $1/2$ to each bifundamental. At a new superconformal fixed point, $U(1)$ global symmetries will mix into the R-symmetry so that the R-charge becomes $\vec{R}^{\text{tot}} = \vec{r} + \vec{R}$ as in (22). If the superpotential has net charge zero under the $U(1)$ that is mixing in, the β-functions for the superpotential couplings will not change. We would therefore like to identify the vector space of $U(1)$ charge assignments $\vec{R}$ under which every closed loop in the quiver is neutral. The

orthogonal complement of this space will describe charge assignments that change the β-functions of superpotential couplings.

This is most easily done in terms of the $|V| \times |E|$ oriented incidence matrix A:

$$A_{ve} = \begin{cases} 1 & \text{if } v \in V \text{ is the final vertex of the edge } e \in E \\ -1 & \text{if } v \in V \text{ is the initial vertex of the edge } e \in E \\ 0 & \text{otherwise} \end{cases} \tag{31}$$

A differs from B in that it encodes the directions of the arrows in the quiver (bifundamentals) with signs. Now consider a charge assignment that takes the form

$$\vec{R} = A^{\mathrm{T}} \mathbf{v} \tag{32}$$

for some $\mathbf{v} \in \mathbb{R}^V$. With this assignment, the charge of a bifundamental e is given by the difference of the values of $\mathbf{v}$ at the final vertex $f(e)$ and the initial vertex $i(e)$. It follows that any gauge invariant operator that forms a closed loop in the quiver will have a vanishing charge, since the charge of the operator will be given by a telescoping sum of the charges of the individual bifundamentals. Thus, any R-charge assignment in the image of A^{T} ($\mathrm{Im}\, A^{\mathrm{T}}$) has no effect on the β-functions of superpotential couplings.

Conversely, suppose an R-charge assignment $\vec{R}$ does not alter the β-function of any closed loop in the quiver. Then we can explicitly construct $\mathbf{v}$ for which $\vec{R} = A^{\mathrm{T}} \mathbf{v}$ in the obvious way: Fix $\mathbf{v}_1 = 0$ for definiteness, and set $\mathbf{v}_{v \neq 1}$ recursively by imposing $\mathbf{v}_{f(e)} = R(e) + \mathbf{v}_{i(e)}$ for all $e \in E$, starting from vertex 1. This procedure could only fail if there were at least two paths from vertex 1 to some vertex v with different R-charges $R(p_1)$ and $R(p_2)$. But then appending any path p from v to vertex 1 would form two different gauge invariant operators with R-charges $R(p_1) + R(p)$ and $R(p_2) + R(p)$, respectively. Because all closed loops in the quiver are assumed to have charge 0 under $\vec{R}$, $R(p_1) - R(p_2)$ must also vanish, so the procedure of finding $\mathbf{v}$ is guaranteed to work.

We conclude that the space of charge assignments which leave the β-functions of superpotential couplings invariant is precisely the image of A^{T}. $U(1)$ charge assignments in this space, known as the *bond space* of the graph, cannot affect the RG running of closed loop operators, and will not play a role in driving a relevant superpotential towards being marginal or irrelevant. In contrast, the orthogonal complement of the bond space contains the $U(1)$ charge assignments that do change the β-functions of the superpotential and can hence be used to satisfy (30). This complement is the kernel of A ($\ker A$), also known as the *cycle space* of the graph.

The dimensions of $\mathrm{Im}\, A^{\mathrm{T}}$ and $\ker A$ are known. This is because $A\, A^{\mathrm{T}} = L$ is the Laplacian matrix of the quiver graph, which is known to have exactly one zero eigenvalue since the quiver is connected [21]. Thus,

$$\dim \mathrm{Im}\, A^{\mathrm{T}} \geq \dim \mathrm{Im}\, L = |V| - 1 \,. \tag{33}$$

On the other hand, $A^{\mathrm{T}}\mathbf{1} = \vec{0}$, which is simply the statement that every edge has an equal number of initial and final vertices. Therefore

$$\dim \operatorname{Im} A^{\mathrm{T}} \leq |V| - 1 \tag{34}$$

and an equality follows. Thus, we have exactly $|V| - 1$ linearly independent charge assignments which do *not* impact the marginality of the superpotential and $|V| + 1$ orthogonal ones that do, i.e.,

$$\dim \operatorname{Im} A^{\mathrm{T}} = |V| - 1 \qquad \text{and} \qquad \dim \ker A = |V| + 1\,. \tag{35}$$

The latter set, of course, contains $\vec{1}$.

4.3   The space of superconformal charge assignments

To check whether one of our quivers has an R-charge assignment consistent with a superconformal fixed point, we can pursue the following strategy: (i) Split the total charge as $\vec{R}^{\mathrm{tot}} = \vec{r} + \vec{R}$ where $\vec{r}$ makes the NSVZ β-functions vanish at the $W = 0$ fixed point; (ii) Choose $\vec{R}$ to have no effect on the NSVZ β-functions while driving the superpotential to marginality.

Recall from Secs. 4.1 and 4.2 that the kernel of the unoriented incidence matrix of the quiver ($\ker B$) encodes charge assignments that leave the NSVZ β-functions invariant, while the charge assignments in the kernel of the oriented incidence matrix ($\ker A$) can be used to drive a relevant superpotential towards being marginal. Thus we are interested in precisely the charge assignments that are contained in the intersection

$$\mathcal{Y} \equiv \ker B \cap \ker A\,; \qquad n \equiv \dim \mathcal{Y}\,. \tag{36}$$

If a solution exists, both the NSVZ β-functions and the marginality inequalities can be satisfied by picking $\vec{R}^{\mathrm{tot}} = \vec{r} + \vec{y}$ where $\vec{y} \in \mathcal{Y}$. But the orthogonal complements of $\mathcal{Y}$ play a role in the physics, too. Specifically, the orthogonal complement of $\mathcal{Y}$ in $\ker A$, $\mathcal{V} \equiv \mathcal{Y}^{\perp} \cap \ker A$, comprises all the remaining charge assignments that change the β-functions of superpotential couplings. This will be of use in analyzing the likely direction of flow of theories that escape to strong coupling like the octahedral example in Sec. 3. For a choice of basis $\{\vec{v}_i\}$ of $\mathcal{V}$, note that i ranges between 0 and $|V| - n$ and choose to set $\vec{v}_0 \equiv \vec{1}$.

Similarly the orthogonal complement of $\mathcal{Y}$ in $\ker B$, $\mathcal{Z} \equiv \mathcal{Y}^{\perp} \cap \ker B$, spans the R-charge assignments that change neither the β-functions for the gauge couplings nor those for the superpotential couplings. Thus, charge vectors $\vec{z} \in \mathcal{Z}$ play no role in showing the *existence* of a superconformal fixed point. However, the physical charge, determined by a-maximization [15] at a superconformal fixed point, may include a component in $\mathcal{Z}$. We note that the dimension of $\mathcal{Z}$ is $|V| - n$ and that in general $\mathcal{V} \not\perp \mathcal{Z}$.

To find a basis for $\mathcal{Y}$, recall that vectors $\vec{y} \in \mathcal{Y}$ are characterized by

$$A\,\vec{y} = B\,\vec{y} = \mathbf{0} \,. \tag{37}$$

Eq. (37) represents $|V|$ identical sets of 2 scalar equations each:

$$p + q - r - s \;=\; 0 \tag{38}$$
$$p + q + r + s \;=\; 0 \,. \tag{39}$$

The variables, which refer to the R-charges of the four bifundamentals incident on a given vertex, are presented in Fig. 7 along with the solution. We see that on each
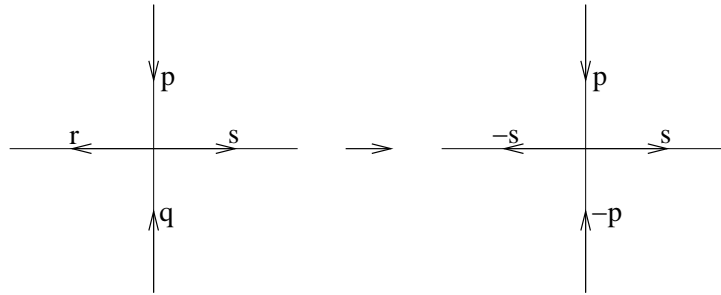

Figure 7: A local solution to equations $A\,\vec{y} = B\,\vec{y} = \mathbf{0}$.

vertex, locally, there are two independent ways of satisfying (37). One of them assigns opposite values to the incoming arrows ($p \neq 0$, $s = 0$), and the other assigns opposite values to outgoing arrows ($p = 0$, $s \neq 0$). Now notice that each of the two modes can be propagated, starting from a single edge assigned a non-zero value, to form a global solution of (37) by the following simple rule:

> At each vertex, if one incoming (outgoing) arrow is assigned a value p, assign the other incoming (outgoing) arrow the value $-p$, leaving the assignments of the outgoing (incoming) arrows unchanged.

The procedure is continued until the resulting sequence of connected edges closes back on itself, as do the dashed lines in Figure 8. After one mode is constructed, one may construct another, starting with any edge not covered by the previously constructed modes. When no uncovered edges remain, one has obtained $n = \dim \mathcal{Y}$ basis vectors $\vec{y}_i$. The modes $\vec{y}_i$ enjoy a number of desirable properties. They are:

- closed lines in the quiver, consisting of alternating $+/-$ signs,

- orthogonal[1] in $\mathbb{R}^E$ (by construction),

- uniquely defined,

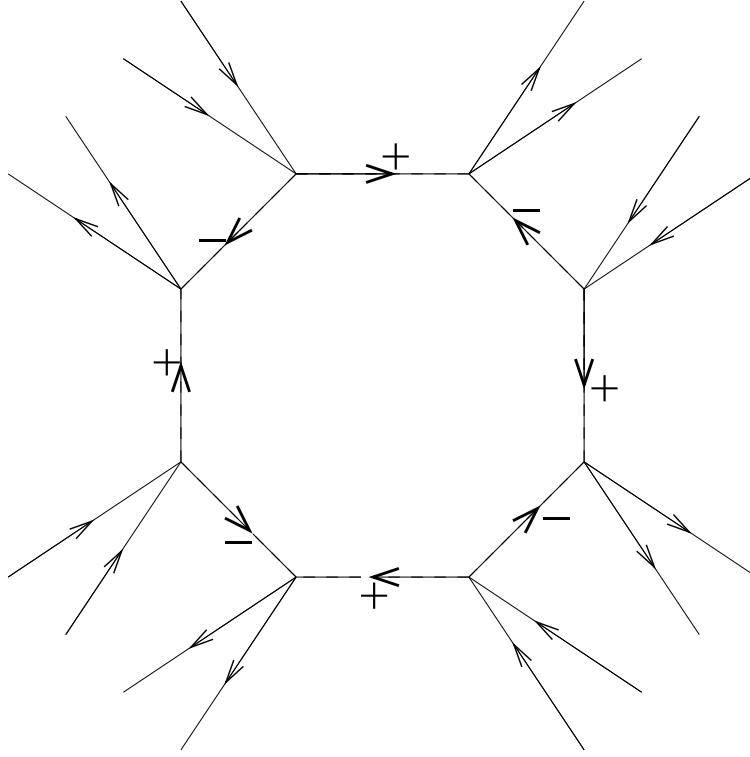[1]Throughout this paper, orthogonality is defined with respect to the Euclidean norm.

Figure 8: A basic circuit $\vec{y}_i$ is marked with dashed lines.

- efficiently constructed (it takes $2|V|$ steps to identify the vectors $\vec{y}_i$);

- they form a partition of the set of edges E of the quiver, and thus

- they form a basis of $\mathcal{Y}$.

Motivated by these properties, we shall call the modes $\vec{y}_i$ **basic circuits** of a quiver. The parameter n, which is the dimension of $\mathcal{Y}$, is simply the number of basic circuits that make up the quiver.

As an illustration of the concept of basic circuits, we re-draw the quiver diagrams considered in Sec. 3, marking the basic circuits with continuous, dashed, and dotted lines. Note that every double edge (two parallel bifundamentals) automatically forms its own basic circuit. The building blocks of Sec. 3.4 provide another convenient illustration: we recognize the straight lines in Fig. 6 as segments of basic circuits.

5   General Analysis of RG Dynamics

A necessary condition for the existence of an interacting superconformal fixed point is that the superpotential does not contain a "forbidden set," i.e. a set of superpotential terms that cannot all simultaneously be marginal or irrelevant without spoiling the
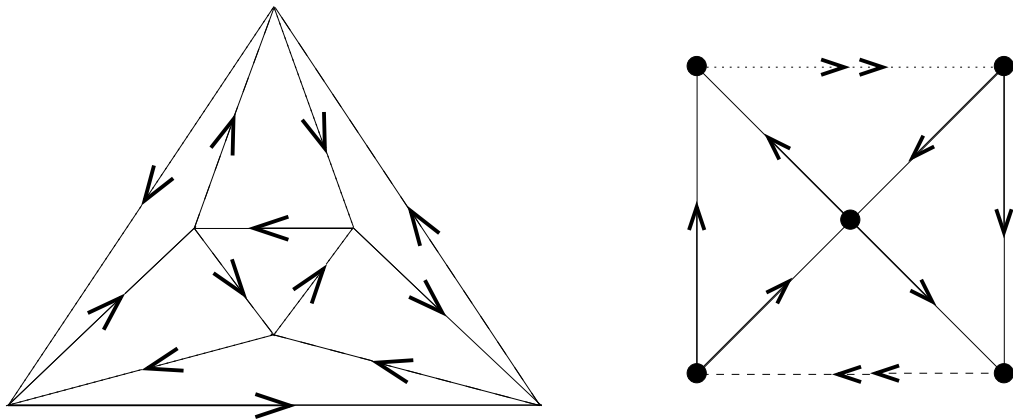
Figure 9: A decomposition of the quivers considered in Sec. 3 into basic circuits marked with continuous, dashed, and dotted lines.

conformal symmetry or unitarity. In our language, such a forbidden set is one whose marginality inequalities are inconsistent with the vanishing of the NSVZ β-functions and unitarity bound.

In this section we completely characterize all the forbidden sets of an arbitrary $N_f = 2N_c$ quiver theory. In most instances this only requires analyzing the marginality inequalities and the NSVZ β-functions, since it is generically difficult to get negative R-charges. In order to explicitly incorporate the unitarity bound in the formalism, it suffices to modify one equation. We point out the necessary modification in a footnote.

5.1 The fixed point of a deformed theory

In this section we will study the possible charge assignments that preserve the NSVZ β-functions but change the superpotential β-functions. These charges are of the form

$$\vec{R} = \sum_i c_i \vec{y}_i \ \in \ \mathcal{Y} \equiv \ker B \cap \ker A, \tag{40}$$

since charges in $\ker B$ do not change the NSVZ β-functions, and charges in $\ker A$ change marginality. These charges measure the departure from the solution $\vec{r} \equiv \frac{1}{2} \cdot \vec{1}$ of the $W = 0$ theory.

Consider deforming a theory by successive additions of relevant operators. Each time a relevant operator $\mathcal{O}$ is added, the vector $\vec{R}$ must respond so as to drive $\mathcal{O}$ to be marginal or irrelevant. Thus, we must satisfy the inequality

$$R^{\text{tot}}(\mathcal{O}) = \vec{R}^{\text{tot}} \cdot \vec{i}_\mathcal{O} = (\vec{r} + \vec{R}) \cdot \vec{i}_\mathcal{O} \geq 2 \,. \tag{41}$$

The vector $\vec{i}_\mathcal{O}$ counts how many times a given bifundamental appears in $\mathcal{O}$. For example, for operators corresponding to minimal loops in the quiver, the entries in $\vec{i}_\mathcal{O}$ are only 0's and 1's.

One may object that (41) need not be an inequality, but an equation, because the flow will only proceed until the relevant operator driving the flow becomes marginal, not beyond. However, if a theory is successively deformed by individual operators $\mathcal{O}_1$, $\mathcal{O}_2$, it may generally happen that the flow induced by $\mathcal{O}_2$ will drive the operator $\mathcal{O}_1$ (which was marginal before the addition of $\mathcal{O}_2$) to become irrelevant. To allow for this possibility, we leave the $\geq$ sign in (41).

Each operator in the superpotential introduces one marginality inequality. A necessary condition for the existence of an interacting superconformal fixed point is that a simultaneous solution to all such inequalities exists. One may then repeat the exercise and deform the theory with another relevant operator. If, after adding some number of relevant operators one by one, a solution to the resulting inequalities fails to exist, the superpotential includes a forbidden set and the theory will exhibit the type of run-away behavior seen in the octahedral theory. In the remainder of this paper we study such situations in detail.

First, we give a necessary and sufficient condition for a set of terms to be a forbidden set. It turns out to involve the basic circuits introduced in Sec. 4.3, which are defined globally on the quiver. Second, we use the machinery of Sec. 4 to study the trajectory along which the theory escapes to strong coupling. This allows us to put forward a natural guess about the correct Seiberg duality frame in which the infrared phase of the theory should be studied.

5.2 Determination of forbidden sets

To begin, it is useful to recast the marginality inequality (41) in terms of the vector $\vec{R} \in \ker B \cap \ker A$. We will be concerned with "excess" dimensions of operators, measured by how far from marginality they are. We define

$$R^{\text{exc}}(\mathcal{O}) \equiv R^{\text{tot}}(\mathcal{O}) - 2, \tag{42}$$

which can be written as

$$R^{\text{exc}}(\mathcal{O}) = (\vec{R} + \vec{r}) \cdot \vec{i}_{\mathcal{O}} - 2 = \sum_{i=1}^{n} c_i \vec{y}_i \cdot \vec{i}_{\mathcal{O}} + p_{\mathcal{O}}. \tag{43}$$

Here we have used (40) and defined $p_{\mathcal{O}}$ to be the excess R-charge of the operator $\mathcal{O}$ above marginality at the $W = 0$ fixed point, where all R-charges are $1/2$. For example, $p_{\mathcal{O}} = -\frac{1}{2}$ for cubic operators, 0 for quartics, $\frac{1}{2}$ for quintics, etc.[2] The physical content of (43) is simply that the net deviation from being marginal is measured by the deviation with all R-charges set equal to $1/2$, plus however much gets added via $\vec{R}$. In terms of (43), the marginality inequalities take the simple form $R^{\text{exc}}(\mathcal{O}) \geq 0$.

---

[2]To incorporate unitarity bounds in a similar way, one introduces an analogue of eq. (43) that measures the "excess" R-charge of a bifundamental above the unitarity bound. It reads $R^{\text{exc}}(e) = \sum_{i=1}^{n} c_i \vec{y}_i \cdot \vec{i}_e + \sum_{i=1}^{|V|-n} b_i \vec{z}_i \cdot \vec{i}_e + p_e \geq 0$, with $\vec{z}_i \in (\mathcal{Y}^\perp \cap \ker B)$ and $p_e = \frac{1}{2}$.

If we add several operators $\mathcal{O}_j$, $R^{\text{exc}}(\mathcal{O}_j) \geq 0$ must hold for each $\mathcal{O}_j$. The inequalities (43) for a set of operators $\mathcal{O}_j$, $j = 1, ..., m$ can be usefully expressed in a matrix form

$$
\begin{pmatrix} c_1 & \cdots & c_n & 1 \end{pmatrix}
\begin{pmatrix} (\vec{y}_1)^{\mathrm{T}} & 0 \\ \vdots & \vdots \\ (\vec{y}_n)^{\mathrm{T}} & 0 \\ (\vec{0})^{\mathrm{T}} & 1 \end{pmatrix}
\begin{pmatrix} \vec{i}_{\mathcal{O}_1} & \vec{i}_{\mathcal{O}_2} & \cdots & \vec{i}_{\mathcal{O}_m} \\ p_{\mathcal{O}_1} & p_{\mathcal{O}_2} & \cdots & p_{\mathcal{O}_m} \end{pmatrix}
\geq \begin{pmatrix} 0 & 0 & \cdots & 0 \end{pmatrix},
$$

(44)

where the inequality should hold for each component. We can rewrite this as

$$
\begin{pmatrix} c_1 & \cdots & c_n & 1 \end{pmatrix} M' \geq \begin{pmatrix} 0 & 0 & \cdots & 0 \end{pmatrix},
$$

(45)

where M' is the product of the two rightmost matrices on the lefthand side of (44). The matrix M' is easy to compute given our choice of basis in terms of basic circuits. Each non-trivial entry in M' is given by $\vec{y}_i \cdot \vec{i}_{\mathcal{O}_j}$, which is simply the (signed) number of edges that the jth operator has in common with the ith basic circuit.

If each inequality in (45) is satisfied, so is any linear combination of these inequalities with non-negative coefficients. The only way in which the marginality inequalities could be inconsistent is if some suitably normalized linear combination of the basic inequalities produces $-1 \geq 0$. To characterize this situation it is convenient to augment the matrix M' to M by adding a leading column,

$$
M = \begin{pmatrix} 0 & \\ \vdots & M' \\ 0 & \\ 1 & \end{pmatrix} = \begin{pmatrix} (\vec{y}_1)^{\mathrm{T}} & 0 \\ \vdots & \vdots \\ (\vec{y}_n)^{\mathrm{T}} & 0 \\ (\vec{0})^{\mathrm{T}} & 1 \end{pmatrix}
\begin{pmatrix} \vec{0} & \vec{i}_{\mathcal{O}_1} & \vec{i}_{\mathcal{O}_2} & \cdots & \vec{i}_{\mathcal{O}_m} \\ 1 & p_{\mathcal{O}_1} & p_{\mathcal{O}_2} & \cdots & p_{\mathcal{O}_m} \end{pmatrix}.
$$

(46)

Replacing M' with M in eq. (45) yields

$$
\begin{pmatrix} c_1 & \cdots & c_n & 1 \end{pmatrix} M \geq \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix},
$$

(47)

and an inconsistent linear combination of inequalities leads to

$$
\begin{pmatrix} c_1 & \cdots & c_n & 1 \end{pmatrix} M \begin{pmatrix} 1 \\ w_1 \\ \vdots \\ w_m \end{pmatrix} = 0,
$$

(48)

for some non-negative coefficients w_j. When a superconformal fixed point does not exist, this equation holds for arbitrary values of c_i, so

$$
M \begin{pmatrix} 1 \\ w_1 \\ \vdots \\ w_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad w_j \geq 0
$$

(49)

necessarily follows from the *non-existence* of a superconformal fixed point solution. This establishes that eq. (49) is a necessary condition a set of terms to be forbidden. The selection of terms forming a forbidden set enters (49) through M.

It is easy to see that eq. (49) is also a sufficient condition, if we accept that any consistent set of R-charges implies the existence of a superconformal fixed point (which is a mild assumption for our $N_f = 2N_c$ quivers). For if a consistent assignment of R-charges exists, then for some choice of c_i eq. (47) has only non-negative entries and the right hand side of (48) is at least 1, so eq. (49) cannot hold. Thus, the part of ker M that lives in the positive orthant[3] defines the forbidden sets of the quiver. We again point out that the non-trivial entries of M are given by the scalar products $\vec{y}_i \cdot \vec{i}_{\mathcal{O}_j}$, which are exceedingly easy to compute.

Summary

We have shown that the inequalities resulting from requiring a set of operators to be marginal or irrelevant can be nicely summed up in a coordinate-independent language. Given a quiver diagram G and a set of operators $\{\mathcal{O}_j\}$ present in the superpotential, one can determine if the set of operators is forbidden via the following decision algorithm:

1. Identify the basic circuits $\vec{y}_i$ of G (Sec. 4.3).

2. Compute the matrix M according to eq. (46). The columns of M correspond to the operators $\mathcal{O}_j$. The non-trivial entries are given by the (signed) number of edges that the basic circuit $\vec{y}_i$ shares with the loop corresponding to $\mathcal{O}_j$

3. Find ker M and check if it intersects the positive orthant, eq. (49).

4. If and only if the answer to the previous step is yes, the set $\{\mathcal{O}_j\}$ is a forbidden set.

5.3 Escape to strong coupling

In the octahedral theory, we noted that adding a forbidden set of operators to the superpotential drives the theory to strong coupling. On general grounds, we expect such behavior since there is no interacting superconformal fixed point from the standpoint of the original degrees of freedom. When a gauge group factor gets strongly coupled, it is natural to perform a Seiberg duality transformation on that node and to rewrite the theory in terms of different degrees of freedom which better describe the IR physics. In this section, we put forward a canonical guess for which node to first dualize.

---

[3]The positive orthant is the set $\{w = (w_i) \,|\, w_i \geq 0, \, i = 1, \ldots, m\}$; after [22].

To arrive at this guess we pursue the following logic. Starting at the $W = 0$ fixed point with the associated superconformal R-charges, we add a "forbidden set" of relevant deformations characterized by their "deficit" of R-charge below marginality. The sign of the β-functions is such that RG flow will drive these deformations towards marginality. Since we cannot dynamically solve the flow equations, we estimate the direction of flow by assuming that it will act to decrease the deficit R-charge of the superpotential deformations in a direction that maximally reduces their mutual incompatibility with marginality. Given this direction of flow for the R-charges, we can analyze the resulting direction of flow of the NSVZ β-functions. It is natural to guess that the group with the steepest NSVZ β-function is the first to become strongly coupled and hence the one to be Seiberg dualized. Since we cannot actually follow the flow to strong coupling, we have no guarantee that the group with the steepest NSVZ β-function at $W = 0$ is really the first to become strongly coupled, so one should regard this as a first approximation.

Specifically, suppose the superpotential contains a forbidden set $\{\mathcal{O}_j\}$. At least some subset of these operators are relevant; for them the quantity $R^{\mathrm{exc}}(\mathcal{O}_j)$ in eq. (43) is negative and represents a deficit of R-charge which must diminish along the flow. If we insist that the β-functions for the gauge couplings vanish, the flow can only alter the R-charges by a vector that lies in $\mathcal{Y} = \ker B \cap \ker A$, as in (40). The precise direction in the space $\mathcal{Y}$ along which $R^{\mathrm{exc}}(\mathcal{O}_j)$ increases towards zero (i.e. toward marginality) most steeply is given by the gradient of $R^{\mathrm{exc}}(\mathcal{O}_j)$ with respect to the coefficients c_i. But because the set $\{\mathcal{O}_j\}$ is forbidden, using (48) we have

$$\frac{\partial}{\partial c_i}\left(\sum_j w_j R^{\mathrm{exc}}(\mathcal{O}_j)\right) = \frac{\partial}{\partial c_i}(-1) = 0 \tag{50}$$

for some choice of $w_j \geq 0$. This confirms that the modes $\vec{y}_i$ cannot simultaneously bring all the operators in $\{\mathcal{O}_j\}$ to marginality, in accordance with the definition of a forbidden set. Said differently, the parenthesis in eq. (50) represents an essential deficit of R-charge which cannot be quenched by any vector in the charge subspace $\mathcal{Y}$. Thus a flow proceeding along a direction in $\mathcal{Y}$ will neither affect the gauge β-functions, nor will it reduce the particular linear combination of charge deficits in (50).

However, we know that the β-functions for the superpotential couplings will drive a flow until this R-charge deficit is quenched. In Sec. 4.2 we saw that the running of the superpotential is affected by shifting R-charges by charge vectors $\vec{R}$ that are drawn from the space $\ker A$. Because the argument in the previous paragraph excludes the space $\mathcal{Y} = \ker B \cap \ker A$, the charge vector $\vec{R}$ must include a shift in a direction contained in the orthogonal complement, $\mathcal{V} = (\mathcal{Y}^\perp \cap \ker A)$. The effect of the modes

of $\mathcal{V}$ on the marginality of operators is captured by an analogue of eq. (43):

$$R_{\mathcal{V}}^{\text{exc}}(\mathcal{O}_j) = \sum_{i=1}^{n} a_i \vec{v}_i \cdot \vec{i}_{\mathcal{O}_j} \tag{51}$$

The deficit of R-charge represented by the parenthesis in (50) will then diminish most steeply in the direction determined by a gradient taken with respect to the coefficients a_i:

$$\frac{\partial}{\partial a_i} \left(\sum_j w_j R_{\mathcal{V}}^{\text{exc}}(\mathcal{O}_j) \right) = \vec{v}_i \cdot \left(\sum_j w_j \vec{i}_{\mathcal{O}_j} \right) \propto \Delta a_i \tag{52}$$

We will assume that the flow proceeds along this line of steepest descent. Then (52) represents the relative intensities by which the respective $\vec{v}_i$ modes will be turned on in the flow.

We now wish to compute how such a change of R-charges will affect the gauge β-functions. Because of eq. (26), the change in the NSVZ β-functions is straightforwardly related to the change in $\vec{R}$ via

$$\Delta \beta_v \sim - \sum_e B_{ve} \Delta R^e. \tag{53}$$

$\Delta \vec{R}$ is easy to compute. It follows from the change in a_i, eq. (52), and is given by

$$\Delta \vec{R} = \sum_i \vec{v}_i \, \Delta a_i \propto \left(\sum_i \vec{v}_i \vec{v}_i^{T} \right) \left(\sum_j w_j \vec{i}_{\mathcal{O}_j} \right) = \mathcal{P}_{\mathcal{V}} \left(\sum_j w_j \vec{i}_{\mathcal{O}_j} \right) , \tag{54}$$

where we have used the projection operation $\mathcal{P}_{\mathcal{V}} = \sum_i |v_i\rangle\langle v_i|$. The appearance of the projection operator onto $\mathcal{V}$ fixes the normalization of $\vec{v}_i$, which was hitherto unspecified. Substituting in (53) gives

$$\Delta \beta_v \propto -B \, \mathcal{P}_{\mathcal{V}} \left(\sum_j w_j \vec{i}_{\mathcal{O}_j} \right) . \tag{55}$$

In the next paragraph we demonstrate that the projector $\mathcal{P}_{\mathcal{V}}$ may in fact be removed from the resulting expression, since the linear combination $\sum_j w_j \vec{i}_{\mathcal{O}_j}$ is already in $\mathcal{V}$.

First, we notice that since each vector $\vec{i}_{\mathcal{O}_j}$ corresponds to a loop in the quiver,

$$A \, \vec{i}_{\mathcal{O}_j} = \mathbf{0} \, , \tag{56}$$

where A is the oriented incidence matrix defined in (31). Therefore, the expression next to $\mathcal{P}_{\mathcal{V}}$ in eq. (55) lies in the kernel of A. On the other hand, because the set $\{\mathcal{O}_j\}$ is forbidden, we know from (49) that the basis vectors of $\mathcal{Y}$ are each orthogonal

to $\sum_j w_j \vec{i}_{\mathcal{O}_j}$. Therefore, $\sum_j w_j \vec{i}_{\mathcal{O}_j}$ lies in that part of $\ker A$ which is orthogonal to $\mathcal{Y}$, that is, $\mathcal{V}$. But then $\mathcal{P}_\mathcal{V}$ is redundant in (55) and we may write

$$\Delta\beta_v \sim -B\left(\sum_j w_j \vec{i}_{\mathcal{O}_j}\right) = -B_{ve}\left(\sum_j w_j i_{\mathcal{O}_j}^e\right). \qquad (57)$$

The resulting expression is a first approximation to the change in the NSVZ β-functions that results from including the forbidden set $\{\mathcal{O}_j\}$ in the superpotential. Accordingly, the most negative entry in (57) defines a canonical guess for the first node to Seiberg dualize.

Notice that the expression on the right hand side of (57) simply counts the number of edges incident on each vertex, weighted loop-wise by the quantities w_j. Therefore, it is manifestly non-positive on each vertex. It follows that a deformation by a relevant operator initially does not push gauge couplings toward zero *if we stick to the original degrees of freedom*. This motivates a statement which we have made previously several times, that including a forbidden set in the superpotential always forces some couplings to become strong. Of course, after following the trajectory to the deep infrared some gauge couplings may flow to zero. The present analysis misses this because (a) it is only an approximation, (b) it only tells you the *initial* direction of flow, and (c) it does not capture the effects which may become apparent in terms of the infrared (Seiberg dual) variables.

Caveats Using eq. (57) to identify the first node which Seiberg dualizes is subject to a number of caveats. First, renormalization group flows are not guaranteed to proceed according to the above computations, since we do not know the details of the strongly coupled physics. While the flow must necessarily head in a direction of decreasing R-charge deficit, we are not aware of a physical principle which would rigorously establish that it always proceeds in the direction of steepest ascent. More importantly, translating an R-charge perturbation (54) into β-functions (57) is only valid if the coefficients of proportionality relating eq. (24) to the full β-function (5) are uniform across all the gauge couplings $g_{v\in V}$, which does not in general hold. That said, eq. (57) is still useful as a rough indicator of the direction of flow, if only because it is extremely easy to compute.

Another subtlety is that eq. (57) may be used only for **minimal forbidden sets**, i.e., those for which there exists only one linear dependence (49). This is because RG trajectories depend on the initial conditions in coupling space, but our formalism only takes into account which operators are present in the superpotential, irrespective of their couplings. The dependence of the flow on the initial conditions must be reflected in eq. (57), and that can only happen via the quantities w_j. When there is only one choice of w_j, it signals that all run-away trajectories resulting from the forbidden set $\{\mathcal{O}_j\}$, *regardless of initial conditions*, asymptote to a particular trajectory defined by

The dashed line of Fig. 2 was an example of such an asymptoting RG flow. We conclude that it is only sensible to use eq. (57) when the forbidden set contained in the superpotential is minimal.

5.4   An example

In this subsection we present an example application of the methods of Secs. 4 and 5. Consider a gauge theory given by the quiver in Fig. 10. It contains two basic circuits,
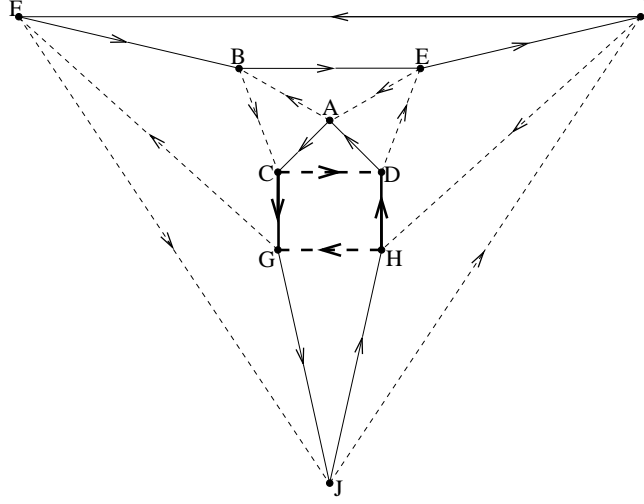

Figure 10: The two basic circuits are marked with thick and thin lines, respectively. Dashed lines carry $-$ signs and continuous lines, $+$ signs.

which are drawn in thick and thin lines, respectively. The edges charged $+1$ (-1) are marked with continuous (dashed) lines.

This theory contains four cubic, seven quartic, and two quintic gauge invariant operators, as may be verified using standard graph theoretic techniques (see [23] for a pedagogical treatment). Using the technology of Sec. 5.2, the matrix M listing the charges of these operators under the modes $\vec{y}_i$ is given by:

$$
M \;=\; \begin{array}{c}
\begin{array}{ccccccccccccc}
\text{\scriptsize ABE} & \text{\scriptsize ACD} & \text{\scriptsize FJI} & \text{\scriptsize GJH} & \text{\scriptsize ABCD} & \text{\scriptsize ACDE} & \text{\scriptsize BEIF} & \text{\scriptsize FJHG} & \text{\scriptsize GJIH} & \text{\scriptsize DEIH} & \text{\scriptsize BCGF} & \text{\scriptsize ABCDE} & \text{\scriptsize FJIHG}
\end{array}\\
\left(\begin{array}{rrrrrrrrrrrrr}
0 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & -1 & -1 & 1 & 1 & -1 \\
0 & -1 & 2 & -1 & 2 & -1 & -1 & 4 & -1 & -1 & -1 & -1 & -4 \\
1 & -\tfrac{1}{2} & -\tfrac{1}{2} & -\tfrac{1}{2} & -\tfrac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2}
\end{array}\right)
\end{array}
\tag{58}
$$

We have marked the columns of M with the operators to which they correspond. The upper (middle) row of M corresponds to the basic circuit marked with thick (thin) lines in Fig. 10. In these rows, the jth entry is simply the number of continuous lines minus the number of dashed lines in a given basic circuit that overlap with $\mathcal{O}_j$.

It is straightforward to find the non-negative coefficient linear dependencies among the columns of M, as dictated by eq. (49). Each of them corresponds to one choice of superpotential terms that forces the theory to run to infinite coupling. When the coefficients w_j are fed into eq. (57), the result selects likely candidates for the fastest-growing couplings and *eo ipso* the first nodes to Seiberg dualize. We list the results in Table 1 below.

Forbidden set	w_j	Max $-\Delta\beta_v$	Forbidden set	w_j	Max $-\Delta\beta_v$
ABE	2	B, E	FJI	2	F, I
BEIF	1/2		BEIF	1/2	
ABE	1	A, D, E	FJI	1	F, G, J
ACD	1		GJH	1	
DEIH	1		BCGF	1	
ABE	1	E, H	FJI	1	C, F
GJH	1		ACD	1	
DEIH	1		BCGF	1	
FJI	1	D, I	ABE	1	B, G
ACD	1		GJH	1	
DEIH	1		BCGF	1	
FJI	1	H, I, J	ABE	1	A, B, C
GJH	1		ACD	1	
DEIH	1		BCGF	1	

Table 1: Individual boxes contain minimal forbidden sets of the theory. Combining terms from different boxes without exhausting any one of them does not produce a forbidden set. The remaining columns list the choices of coefficients w_j which satisfy eq. (49) and the nodes with the most negative value of $\Delta\beta_v$ in eq. (57).

6 Discussion

This paper has explored a large class of quiver theories, within which the endpoints of renormalization group flows depend directly on the topology of the associated quiver diagrams. Specifically, we found that when an $SU(N_c)^{|V|}$ quiver with $N_f = 2N_c$ at all nodes is deformed away from the $W = 0$ fixed point by a set of relevant operators, the ensuing flow runs away to infinite values of the couplings if and only if the operators are chosen in such a way that (49) is satisfied for some choice of $w_j \geq 0$.

The condition (49) depends on the basic circuits of the quiver diagram, which are closed sequences of edges in the quiver with alternating signs attached to them (see Sec. 4.3 for a precise definition). The number n of basic circuits in a quiver is a heuristic measure of how flexible the theory is at accommodating successive relevant deformations without losing conformality. When n is large, the theory has more freedom to adjust its R-charges without changing the beta functions of its gauge

couplings, so as to render all relevant operators marginal. On the other hand, when n is small, a generic perturbation is likely to produce a trajectory bound for strong coupling. Because basic circuits form a partition of the edge set of a quiver graph, we always have $n \geq 1$. At the other extreme, n is bounded from above by $|V|$. This bound is saturated by cyclic quiver diagrams whose edges always appear in pairs.

It would be useful to broaden our results to a larger class of quiver theories. For example, if we wish to fully understand the IR limits of quivers of the type we have considered, we must also learn how to deal with the more general class of quivers they can flow to. For example, nodes of equal-rank quivers will change rank when the RG flow ventures onto a Higgs branch via giving VEVs to some of the bifundamentals. Additionally, Seiberg dualizing one node can change the number of flavors on adjacent nodes, thereby taking the quiver away from $N_f = 2N_c$.

It would also be interesting to understand how the results in this paper manifest themselves in the tiling picture of [24]. For example, it seems possible that since forbidden sets drive flows to infinite coupling, there might be no consistent way of dualizing a quiver with a forbidden superpotential so as to give a tiling. This observation may be related to the results of [25].

Extending our formalism to broader classes of quivers will require some thought since the concept of a basic circuit is only well-defined for our class. As a first step, one could consider quivers where all nodes are $SU(N_c)$ except one which is $SU(N_c+1)$. A simple quiver of this sort is the deformed conifold, where the addition of M fractional 3-branes changes the gauge group from $SU(N) \times SU(N)$ to $SU(N + M) \times SU(N)$, and leads to the celebrated Klebanov-Strassler duality cascade [9]. More complex quivers with unequal ranks are believed (based on a node-by-node analysis) to admit chaotically cascading flows [10] and other exotic RG dynamics [11]. It would be remarkable if such complex behaviors could be deduced from the topology of the underlying quiver diagram.

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